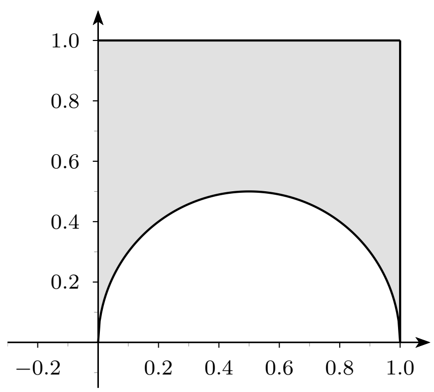
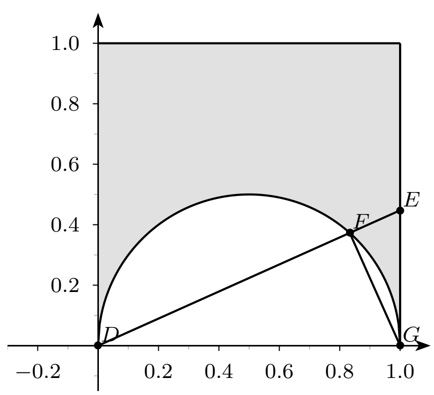
MATH2020A Tutorial 3

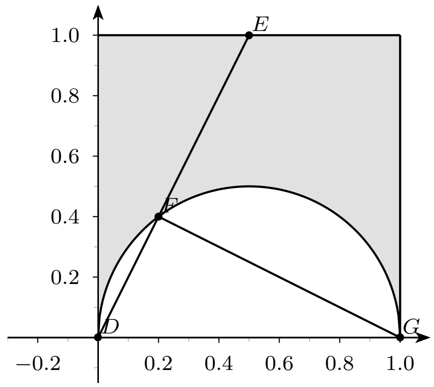
1. Describe the given region in polar coordinates



Solution. First we draw the following lines,



So we find $DF = \cos\theta, DE = \frac{1}{\cos\theta}$. Hence we have $\cos\theta \leq r \leq \frac{1}{\cos\theta}$. For another part,



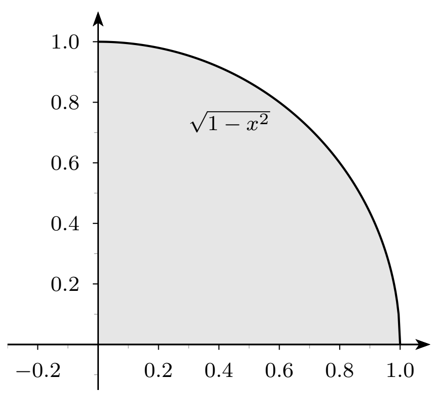
We have $DE = \frac{1}{\cos(\frac{\pi}{2}-\theta)} = \frac{1}{\sin\theta}$. So we have $\cos\theta \le r \le \frac{1}{\sin\theta}$. Combining them, we get

$$0 \le \theta \le \frac{\pi}{4}, \cos\theta \le r \le \frac{1}{\cos\theta} \text{ or } \frac{\pi}{4} \le \theta \le \frac{\pi}{2}, \cos\theta \le r \le \frac{1}{\sin\theta}$$

2. Evaluating the following integration.

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{1+\sqrt{x^2+y^2}} dydx$$

Solution. Sketch this region.



Based on this region, we might want to use polar coordinate.

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{1+r} r dr d\theta = \int_0^{\frac{\pi}{2}} \left[r - \ln(1+r)\right]_0^1 d\theta \\ &= \int_0^{\frac{\pi}{2}} (1-\ln 2) d\theta = \frac{\pi}{2} - \frac{\pi}{2}\ln 2 \end{aligned}$$

3. Evaluating the following integration.

$$I = \int_0^{\frac{\pi}{4}} \int_0^{\sec y} x^2 \cos y dx dy$$

Solution, you can integrate directly, says

$$I = \int_0^{\frac{\pi}{4}} \frac{\sec^2 y}{3} dy = \frac{1}{3} [\tan y]_0^{\frac{\pi}{4}} = \frac{1}{3}$$

But there are also another way to do it. This form is like a polar coordinate, so we write it as

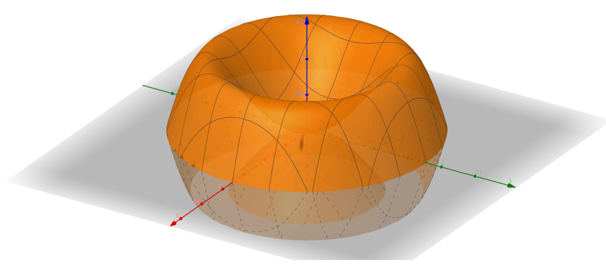
$$I = \int_0^{\frac{\pi}{4}} \int_0^{\sec \theta} r^2 \cos \theta dr d\theta$$

And then change polar coordinate to the usual coordiante and get

$$I = \int_0^1 \int_0^x x dy dx = \int_0^1 x^2 dx = \frac{1}{3}$$

This integration is relatively easier than the previous one.

4. Find the volume between the surface $z = \sin(x^2 + y^2)$ and $z = -\sin(x^2 + y^2)$ with $x^2 + y^2 \leq \pi$.



Solution.

$$\begin{aligned} V &= \iiint_R dx dy dz = \iint_{\{x^2+y^2 \leq \pi\}} \int_{-\sin(x^2+y^2)}^{\sin(x^2+y^2)} dz dx dy \\ &= \iint_{\{x^2+y^2 \leq \pi\}} 2 \sin(x^2 + y^2) dx dy = \int_0^{2\pi} \int_0^{\sqrt{\pi}} 2 \sin(r^2) r dr d\theta \\ &= \int_0^{2\pi} \left[\cos(r^2)\right]_0^{\sqrt{\pi}} d\theta = 4\pi \end{aligned}$$

5. Evaluation the following integration.

$$I = \int_0^1 \int_z^1 \int_y^1 \frac{e^{x^2}}{y} dxdydz$$

Remark. Notice this integration might not well defined in the usual sense at $y = 0$. But here we just assume we can change the integration order as we want.

Solution. Change the order of integration, we get

$$\begin{aligned} I &= \int_0^1 \int_0^x \int_0^y \frac{e^{x^2}}{y} dzdydx = \int_0^1 \int_0^x e^{x^2} dydx \\ &= \int_0^1 xe^{x^2} dx = \frac{e-1}{2} \end{aligned}$$